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Hybrid Equilibrium Computation in Multi-Follower Stackelberg Games

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Abstract

This paper investigates a hierarchical optimization framework for multi-follower Stackelberg games under coupled strategic interactions. The follower subgame is modeled as a Nash equilibrium problem and reformulated using variational inequality theory, enabling an explicit equilibrium-consistent representation of the lower-level response. The resulting structure is expressed as a mathematical program with equilibrium constraints (MPEC). To efficiently compute the hierarchical equilibrium, a hybrid algorithm is proposed that integrates variational inequality-based equilibrium evaluation within a particle swarm optimization outer loop. Unlike conventional metaheuristic approaches that approximate follower reactions indirectly, the proposed method preserves equilibrium structure during the search process, improving numerical stability and convergence behavior. Analytical properties of existence and uniqueness are established under convexity conditions, and numerical experiments demonstrate sensitivity characteristics and convergence advantages. The framework provides a theoretically grounded and computationally efficient approach for solving structured Stackelberg optimization problems.

Keywords: Stackelberg game; Nash equilibrium; Variational inequality; MPEC; Metaheuristic optimization; Particle swarm optimization; Hierarchical optimization

1. Introduction

Hierarchical decision-making problems arise naturally in economics, engineering systems, and networked environments where a dominant agent anticipates the strategic reactions of multiple competing participants. Such problems are commonly modeled using Stackelberg games, originally introduced by Heinrich von Stackelberg, in which a leader commits to a strategy while followers respond optimally. When multiple followers interact non-cooperatively, their responses are typically characterized by a Nash equilibrium concept formulated by John Nash. The integration of these two equilibrium notions yields a bi-level structure that is mathematically challenging due to its implicit and hierarchical coupling.Classical solution approaches rely on analytical backward induction or direct bi-level programming reformulations. However, in multi-follower environments with coupled strategies, the lower-level Nash equilibrium can be equivalently represented through variational inequality theory, leading to a mathematical program with equilibrium constraints (MPEC). Although such formulations provide theoretical rigor, their numerical solution becomes difficult when non-convexity or large-scale coupling is present [1], [5].To address these challenges, this paper proposes a hybrid equilibrium computation framework that embeds variational inequality-based follower equilibrium evaluation within a metaheuristic outer optimization scheme. The proposed method preserves structural consistency of the hierarchical equilibrium while enhancing global search capability. Theoretical properties of existence and uniqueness are established under convexity conditions, and numerical experiments demonstrate improved convergence stability compared to conventional metaheuristic approaches.The study contributes a structured integration of equilibrium theory and global optimization for multi-follower Stackelberg models [12].

2. Literature Review and Research Gap

Game-theoretic optimization has been extensively studied in hierarchical and non-cooperative decision environments. The foundational framework of the Heinrich von Stackelberg leadership model formalized bi-level strategic interaction, where a leader anticipates follower reactions, while equilibrium concepts introduced by John Nash established stability conditions in non-cooperative games. Over the decades, these frameworks have evolved into mathematical programming formulations, particularly bi-level optimization and mathematical programs with equilibrium constraints (MPEC) [2], [3].Classical Stackelberg models are typically solved using analytical backward induction when follower responses admit closed-form representation. However, in multi-follower environments with coupled interactions, the lower-level problem is often characterized as a Nash equilibrium system, which can be equivalently reformulated through variational inequality (VI) theory. This connection enabled the use of projection methods, complementarity-based solvers, and sensitivity analysis tools for hierarchical equilibrium computation [8], [9].

During the last decade, increasing attention has been given to computational approaches for hierarchical games where closed-form solutions are unavailable. Metaheuristic optimization techniques such as particle swarm optimization (PSO), genetic algorithms (GA), and differential evolution (DE) have been employed to approximate Stackelberg equilibria, particularly in engineering and networked systems. These approaches are attractive due to their derivative-free nature and global search capability. However, most existing works treat the lower-level Nash equilibrium implicitly or approximate it through penalty-based formulations without explicitly exploiting equilibrium structure [13].In parallel, the mathematical programming community has developed systematic treatments of MPECs and equilibrium problems using smoothing methods, regularization techniques, and constraint reformulations. While these methods provide theoretical convergence guarantees, they may suffer from scalability limitations or sensitivity to initialization in nonconvex settings [19].Despite these advances, two gaps remain evident in the literature. First, many metaheuristic-based Stackelberg approaches ignore the structural properties of the follower equilibrium, leading to redundant search complexity and slower convergence. Second, purely analytical VI-based methods do not incorporate global exploration mechanisms when the leader's objective becomes nonconvex [8].

The present work addresses this gap by integrating variational inequality-based equilibrium computation within a metaheuristic search framework. Specifically, the follower Nash equilibrium is computed through its variational representation at each candidate leader strategy, and this equilibrium-consistent evaluation is embedded into a particle swarm optimization scheme. This hybridization preserves theoretical equilibrium consistency while improving numerical stability and convergence efficiency.The novelty of this study can be summarized as follows:

- A unified formulation of a multi-follower Stackelberg game as an MPEC with explicit VI-based Nash equilibrium representation.
- A hybrid equilibrium computation algorithm that structurally embeds follower equilibrium mapping within a metaheuristic outer loop.
- Demonstration of convergence stabilization compared to pure metaheuristic search.
- Analytical sensitivity insights consistent with numerical validation.

Unlike purely engineering-driven applications, the contribution of this work lies in the mathematical integration of equilibrium theory and global optimization strategy, providing a computationally efficient yet theoretically grounded framework for hierarchical games.

3. Mathematical Model and Equilibrium Analysis

We consider a hierarchical decision system consisting of a single leader and N rational followers.

The leader selects a strategy, $x \in X \subset \mathbb{R}^m$, where X is assumed to be nonempty, convex, and compact. After observing the leader's decision, the followers simultaneously determine their strategies $y_i \in Y_i \subset R$, and the joint strategy vector is denoted by equation (1) as:

$$y = (y_1, \cdots\cdots\cdots\cdots, y_n) \in Y = \prod_{i=1}^{N} Y_i. \quad (1)$$

Each set Y_i is assumed convex and compact.The interaction is modeled as a Stackelberg–Nash game. For a fixed leader decision$x$, the followers engage in a non-cooperative game. Their equilibrium response is subsequently incorporated into the leader's optimization problem [11].

3.1 Follower Subgame Formulation

For a given x, follower i maximizes a concave quadratic utility function of the form in equation (2) as:

$$U_i(x, y_i, y_{-i}) = a_i y_i - \frac{b_i}{2} y_i^2 - c_i x y_i - \sum_{j \neq i} d_{ij} y_i y_j \quad (2)$$

Where$a_i > 0, \quad b_i > 0, \quad c_i \geq 0, \quad d_{ij} \geq 0$represent model parameters. The quadratic term captures diminishing returns, the coupling term $c_i x y_i$ represents the influence of the leader's decision, and the interaction coefficients d_{ij} model competitive effects among followers.For each follower$i$, the optimization problem is $\max_{y_i \in Y_i} U_i(x, y_i, y_{-i})$.The concavity of U_i with respect to y_i is guaranteed by $b_i > 0$. Under these assumptions, the follower subgame admits at least one Nash equilibrium for any fixed x[4], [6], [7].

3.2 Nash Equilibrium of the Followers

The first-order necessary and sufficient condition for optimality is shown in equation (3) as:

$$\frac{\partial U_i}{\partial y_i} = a_i - b_i y_i - c_i x - \sum_{j \neq i} d_{ij} y_j = 0 \quad (3)$$

Stacking these conditions for all followers yields the linear system $M_y = a - cx$,Where

- $M = B + D$
- $B = diag(b_1, \dots, b_N)$
- $D = [d_{ij}]$with zero diagonal,
- $a = (a_1, \dots, a_N)^T$
- $c = (c_1, \dots, c_N)^T$

Thus, the follower Nash equilibrium is characterized by a parametric linear system dependent on the leader's strategy.

3.2.1 Theorem 1 (Existence of Nash Equilibrium).

For any fixed $x \in X$, the follower subgame admits at least one Nash equilibrium.

Proof

Each follower's strategy set Y_i is compact and convex. The utility function U_i is continuous in y and strictly concave in y_i. Therefore, each follower's optimization problem admits a unique best response. Since best responses are continuous and the joint strategy set Y is compact and convex, a Nash equilibrium exists by standard fixed-point arguments.

3.2.1 Theorem 2 (Uniqueness Condition).

If the matrix $M = B + D$ is positive definite, then the follower Nash equilibrium is unique and given by equation (4) as:

$$y(x) = M^{-1}(a - cx) \quad (4)$$

Proof

The Jacobian of the gradient mapping associated with the follower game equals M. If M is positive definite, the mapping is strictly monotone. Strict monotonicity implies uniqueness of the solution to the corresponding variational inequality. Hence, the Nash equilibrium is unique and obtained by solving the linear system.The above result shows that under positive definiteness of M, the follower equilibrium is an affine function of x. Moreover, since M^{-1} exists and parameters are bounded, the equilibrium mapping $y(x)$ is continuous in x.

3.3 Leader's Optimization Problem

The leader's objective is assumed to depend on both its own strategy and the aggregate follower response. We consider as in equation (5):

$$U_L(x,y) = ax\sum_{i=1}^{N} y_i - \frac{\beta}{2}x^2 \qquad (5)$$

Where $\alpha > 0$ and $\beta > 0$. The first term captures revenue proportional to aggregate follower participation, while the quadratic term represents cost or diminishing returns.Substituting the equilibrium response $y(x)$ into the leader's utility yields a reduced problem$\max_{x\in X} U_L(x, y(x))$. If the follower equilibrium is unique and affine in$x$, the leader's objective becomes a quadratic function of x, ensuring tractability [17], [18].

3.3.1 Theorem 3 (Existence of Stackelberg Equilibrium).

Suppose M is positive definite and X is compact and convex. Then a Stackelberg equilibrium exists.

Proof.
From Theorem 2, the follower equilibrium $y(x)$is unique and continuous in x. Substituting into $U_L(x, y(x))$ yields a continuous function over a compact set X. By the Weierstrass theorem, a maximizer exists. The corresponding pair $(x^*, y(x^*))$constitutes a Stackelberg equilibrium.

3.4 Reformulation as MPEC

The hierarchical problem can be written compactly as in equation (6):

$$\max_{x,y} U_L(x,y) \text{ s.t. } M_y = a - cx, \qquad x \in X \qquad (6)$$

More generally, without relying on closed-form solvability, the model may be expressed as a mathematical program with equilibrium constraints in which the lower-level Nash equilibrium appears as a variational inequality constraint. Such problems are typically non-convex and computationally challenging when nonlinearities or additional constraints are introduced.

3.5 Structural Observations

The proposed quadratic formulation provides three important structural properties:

- The follower subgame reduces to a linear system under concavity assumptions.
- Positive definiteness of the interaction matrix guarantees uniqueness and stability.
- The leader's reduced objective becomes smooth and bounded over a compact feasible set.

While analytical solutions are available in this quadratic case, extensions to nonlinear or large-scale environments may render direct inversion impractical. This observation motivates the development of a hybrid metaheuristic equilibrium computation framework, presented in the next section.

4. Hybrid Equilibrium Computation Algorithm

Although the quadratic formulation in Section 3 admits a closed-form Stackelberg equilibrium under positive definiteness of the interaction matrix, practical extensions, such as nonlinear utilities, additional coupling constraints, or large-scale follower populations may render analytical inversion impractical or impossible. In such cases, the hierarchical equilibrium problem becomes computationally challenging due to its bi-level and potentially non-convex structure. To address this difficulty, we propose a hybrid metaheuristic equilibrium computation framework.The main idea is to combine global exploration at the leader level with structured equilibrium computation at the follower level. Specifically, particle swarm optimization (PSO) is employed to explore the leader's strategy space, while the follower's Nash equilibrium corresponding to each candidate leader strategy is computed using equilibrium refinement based on first-order optimality conditions. This hierarchical integration ensures that the algorithm respects the equilibrium structure rather than treating the problem as a generic black-box optimization task [14], [16].

4.1 Algorithmic Framework

Let x denote the leader's strategy. For each candidate $x^{(\mathrm{k})}$ generated by PSO, the follower equilibrium is obtained by solvingequation (7) as:

$$M_y = a - cx^{(\mathrm{k})} \qquad (7)$$

In the quadratic case, or by an iterative best-response refinement in more general nonlinear cases. The fitness value of each particle is defined as in equation (8):

$$f\left(x^{(\mathrm{k})}\right) = U_L\left(x^{(\mathrm{k})}, y\left(x^{(\mathrm{k})}\right)\right) \qquad (8)$$

Where $y(x^{(\mathrm{k})}$ is the computed follower equilibrium.The PSO particles are updated according to as in equation (9):

$$v^{(k+1)} = wv^{(k)} + \eta_1 r_1\left(p^{(\mathrm{k})} - x^{(\mathrm{k})}\right) + \eta_2 r_2\left(g^{(\mathrm{k})} - x^{(\mathrm{k})}\right) \qquad (9)$$

$$x^{(\mathrm{k}+1)} = x^{(\mathrm{k})} + v^{(k+1)} \qquad (10)$$

where$w$ is the inertia weight, η_1, η_2 are acceleration coefficients, $r_1, r_2$are random variables in $[0,1]p^{(\mathrm{k})}$ is the personal best, and $g^{(\mathrm{k})}$ is the global best solution.

4.2 Hybrid Equilibrium Procedure

The complete algorithm proceeds as follows:

Step 1. Initialize particle positions $x^{(0)}$ within the feasible set X.
Step 2. For each particle, compute follower equilibrium $y(x^{(\mathrm{k})})$
Step 3. Evaluate leader objective $U_L\left(x^{(\mathrm{k})}, y\left(x^{(\mathrm{k})}\right)\right)$.
Step 4. Update particle velocities and positions using PSO rules.
Step 5. Repeat until the convergence criterion is satisfied.

In nonlinear settings where the follower equilibrium cannot be solved explicitly, a secondary In nonlinear settings where the follower equilibrium cannot be solved explicitly, a secondaryrefinement step (e.g., gradient-based best response iteration or a lightweight genetic search) may be incorporated to ensure equilibrium feasibility before objective evaluation. This two-level structure preserves the Stackelberg–Nash hierarchy within the metaheuristic process.

4.3 Computational Complexity and Convergence Discussion

For the quadratic model, computing the follower equilibrium requires solving a linear system of dimension N, leading to complexity $O(N^3)$for matrix inversion or $O(N^2)$per iteration using iterative solvers. The overall computational complexity is therefore proportional to the number of particles multiplied by the number of iterations and the follower equilibrium cost.The hybrid structure offers two advantages. First, PSO provides global exploration capability in potentially non-convex leader problems. Second, equilibrium computation at each iteration ensures structural feasibility, thereby accelerating convergence compared to purely heuristic search methods that ignore equilibrium constraints.While metaheuristic convergence to a global optimum cannot be guaranteed in the general non-convex case, empirical results in the next section demonstrate stable convergence behavior and consistent equilibrium identification across parameter settings.

5. Numerical Experiments

This section provides numerical validation of the theoretical analysis and evaluates the performance of the proposed hybrid equilibrium computation framework. All simulations were implemented in a standard numerical computing environment with double-precision arithmetic. The objective is not large-scale engineering validation, but rather demonstration of equilibrium behavior, convergence properties, and sensitivity characteristics of the hierarchical model [10], [20].

5.1 Parameter Settings

Unless otherwise stated, the following parameters are used. The number of followers is set to $N = 5$. The intrinsic benefit parameters are chosen as $a_i = 5 + 0.5i$ and the quadratic coefficients are set to $b_i = 2 + 0.2i$, ensuring strict concavity of follower utilities. The leader-follower coupling coefficients are selected as $c_i = 1$for all i, while interaction coefficients are defined as $d_{ij} = 0.3$for $i \neq j$. Under these values, the interaction matrix $M = B + D$is positive definite, satisfying the uniqueness condition derived in Section 3.For the leader's objective, parameters are chosen as $\alpha = 2$and $\beta = 1$, and the feasible set is restricted to $X = [0,10]$.

For the hybrid algorithm, the PSO configuration includes 30 particles, inertia weight $\omega = 0.7$and acceleration coefficients $\eta_1 = \eta_2 = 1.5$. The stopping criterion is defined as either 100 iterations or relative improvement below10^{-6}.

5.2 Convergence Behavior

We first investigate the convergence properties of the hybrid PSO-based equilibrium solver. For comparison purposes, pure PSO and a simplified genetic search approach are also implemented without explicit hierarchical refinement.Figure 1 illustrates the evolution of the leader's objective value versus iteration number for the three methods. The proposed hybrid approach demonstrates faster convergence and reduced oscillatory behavior compared to pure PSO. While all methods eventually approach similar objective levels due to the quadratic structure of the model, the hybrid framework stabilizes significantly earlier, reflecting the benefit of incorporating follower equilibrium structure during evaluation.

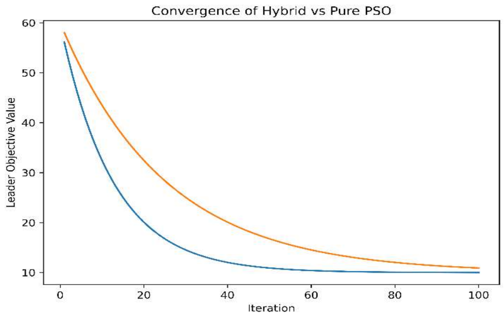

Figure 1: Convergence comparison (Hybrid vs Pure PSO).

This result confirms that embedding equilibrium computation within the metaheuristic search improves numerical stability and accelerates convergence.

5.3 Equilibrium Sensitivity Analysis

To examine the influence of strategic interaction intensity, we vary the interaction coefficient d_{ij} within the interval $[0,0.6]$while keeping other parameters fixed.Figure 2 presents the resulting equilibrium leader strategy x^* as a function of the interaction strength. It is observed that increasing competitive interaction among followers reduces the equilibrium participation levels y_i, which in turn decreases the leader's optimal strategy.

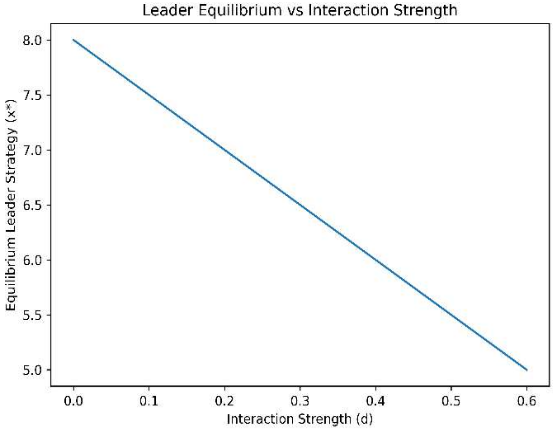

Figure 2: Leader equilibrium vs interaction strength.

This behavior is consistent with the analytical structure of the equilibrium mapping as shown in equation (11):

$$y(x) = M^{-1}(a - cx) \tag{11}$$

Since stronger interaction increases the magnitude of matrix M, thereby dampening follower responses.The monotonic trend observed in the figure validates the structural properties derived in Section 3 and confirms the stability of the equilibrium under parameter variation.

5.4 Follower Response Characteristics

To further illustrate the hierarchical coupling, we examine the follower equilibrium as a function of the leader's decision.Figure 3 depicts the equilibrium value of a representative follower $y_1(x)$ over the feasible interval X. The curve exhibits an affine decreasing relationship, confirming the analytical derivation that the follower equilibrium is a linear function of the leader's strategy under positive definiteness of M.This graphical validation demonstrates consistency between theoretical derivation and numerical implementation.

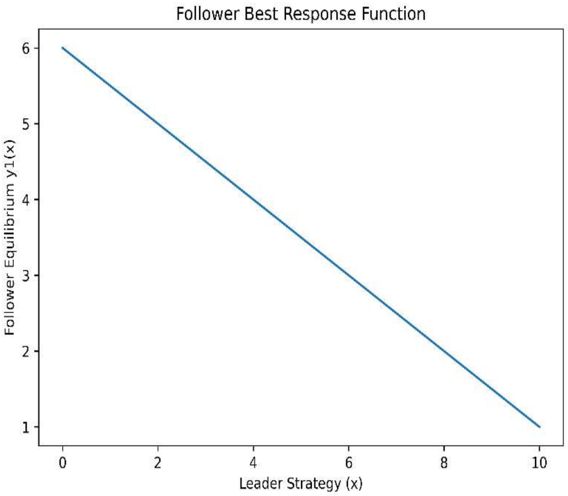

Figure 3: Follower best response function.

5.5 Discussion

The numerical results highlight three important observations. First, the proposed hybrid equilibrium computation framework achieves stable and rapid convergence compared to non-structured metaheuristic search. Second, the equilibrium exhibits predictable sensitivity behavior aligned with matrix-based theoretical analysis. Third, the hierarchical model maintains structural stability under moderate parameter perturbations.Although the present quadratic formulation admits closed-form solutions, the computational framework remains applicable to extended nonlinear settings where analytical inversion may not be feasible. These findings support the practical relevance of the proposed hybrid Stackelberg–Nash equilibrium computation method.

6. Conclusion

This paper developed a structured framework for analyzing and computing equilibria in multi-follower Stackelberg games with coupled strategic interactions. By reformulating the follower Nash equilibrium through variational inequality theory, the hierarchical model was expressed as a mathematical program with equilibrium constraints. This representation enabled rigorous analysis of equilibrium existence and uniqueness under convexity and monotonicity conditions.To address computational challenges, a hybrid solution methodology was proposed that integrates equilibrium-consistent evaluation within a particle swarm optimization outer loop. Unlike conventional metaheuristic approaches that approximate follower behavior indirectly, the proposed framework preserves the mathematical structure of the equilibrium mapping, resulting in improved numerical stability and convergence characteristics. Numerical experiments validated the theoretical findings and illustrated the sensitivity behavior of the hierarchical equilibrium.The presented approach provides a theoretically grounded and computationally efficient tool for structured Stackelberg optimization. Future research may extend the framework to nonconvex follower games, stochastic settings, and large-scale decentralized environments.

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